Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction

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Abstract

The generalized Drinfel'd-Sokolov hierarchies studied by de Groot-Hollowood-Miramontes are extended from the viewpoint of Sato-Wilson dressing method. In the $A_1^{(1)}$ case, we obtain the hierarchy that include the derivative nonlinear Schrödinger equation. We give two types of affine Weyl group symmetry of the hierarchy based on the Gauss decomposition of the $A_1^{(1)}$ affine Lie group. The fourth Painlevé equation and their Weyl group symmetry are obtained as a similarity reduction. We also clarify the connection between these systems and monodromy preserving deformations.

Contents

1 Introduction

There are many brilliant works on the relation between Lie algebras and soliton equations. Among those works, the approach due to Drinfel'd and Sokolov [DS] is a milestone, and gives a method for classifying many soliton equations. Although extended version of their work has been proposed [dGHM, HM, BtK2], there still exist several soliton equations that are not treated along the line of Drinfel'd-Sokolov's works. One example of those equations is a derivative nonlinear Schödinger (∂NLS) equation:

$$\mathrm{i}q_T = \frac{1}{2}q_{XX} + 2\mathrm{i}q^2\bar{q}_X + 4|q|^4 q, \tag{1.1}$$

which has been studied by several authors [ARS, GI, OS, T]. This integrable equation is a modification of the nonlinear Schrödinger (NLS) equation:

$$\mathrm{i}q_T = \frac{1}{2}q_{XX} + 4|q|^2 q. \tag{1.2}$$

Hereafter we will forget the complex structure of (1.1), (1.2) and consider nonlinear coupled equations,

$$\begin{cases} q_t = \dfrac{1}{2}q_{xx} - 2q^2 r_x - 4q^3 r^2, \\ r_t = -\dfrac{1}{2}r_{xx} - 2r^2 q_x + 4r^3 q^2, \end{cases} \tag{1.3}$$

and

$$\begin{cases} q_t = \dfrac{1}{2}q_{xx} + 4q^2 r, \\ r_t = -\dfrac{1}{2}r_{xx} - 4qr^2. \end{cases} \tag{1.4}$$

We note that (1.3), (1.4) is reduced to (1.1), (1.2), respectively, under the condition $r = \bar{q}$, $X = \mathrm{i}x$, $T = \mathrm{i}t$. It is well-known that the hierarchy of soliton equations including NLS (1.4) is obtained as a Drinfel'd-Sokolov hierarchy of $A_1^{(1)}$ homogeneous type.

The aim of the present article is threefold:

Extension of the Drinfel'd-Sokolov formulation

We extend the generalized Drinfel'd-Sokolov hierarchy [dGHM] from the viewpoint of Sato-Wilson dressing method. The extended version includes the ∂NLS equation (1.3) as an $A_1^{(1)}$ case.

Description of affine Weyl group symmetry

There exist transformations of the ∂NLS equation, called Bäcklund transformations that relate two solutions of the ∂NLS equation. We construct two types of Bäcklund transformations that satisfy the relation of the $A_1^{(1)}$ affine Weyl group. We remark that our construction of the affine Weyl group symmetry is an extention of the work by Noumi and Yamada [NY1, NY2].

Algebraic description of similarity reduction

An interesting feature of the ∂NLS equation (1.1) is its connection to the fourth Painlevé equation ($\mathrm{P_{IV}}$):

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \nu_1)y + \frac{\nu_2}{y}, \tag{1.5}$$

where $\nu_1, \nu_2 \in \mathbb{C}$ are parameters. In [ARS], Ablowitz, Ramani and Segur have shown that the self-similar solutions of ∂NLS satisfy $\mathrm{P_{IV}}$ with a special case of the parameters. We give a systematic framework of similarity reductions of the ∂NLS hierarchy that gives $\mathrm{P_{IV}}$ with full-parameters. We also give a relation to monodromy preserving deformation studied by Jimbo, Miwa and Ueno [JMU, JM1, JM2].

As for an application to discrete integrable systems, we consider a discrete equation,

$$X_{n-1} + X_n + X_{n+1} = x + \frac{\kappa_1 n + \kappa_2 + \kappa_3(-1)^n}{X_n}. \tag{1.6}$$

This equation called the asymmetric discrete Painlevé I ($\mathrm{dP_I}$) because a continuous limit of (1.6) with $\kappa_3 = 0$ is the first Painlevé equation. Grammaticos and Ramani[GR] obtained the equation (1.6) from the Schlesinger transformations, which are special type of Bäcklund transformations. We construct a Schlesinger transformations of the ∂NLS equation as an extension of affine Weyl group symmetry and obtain $\mathrm{dP_I}$.

The equations, NLS (1.4), $\mathrm{P_{IV}}$ (1.5), and $\mathrm{dP_I}$ (1.6), share a class of rational solutions expressed by the Hermite polynomials [IY, NY2, OKS]. We clarify the algebraic structure of this class of solutions by using the fermionic representation of $\widehat{\mathfrak{sl}}_2$.

2 Construction of the ∂NLS hierarchy

2.1 General framework

In this subsection, we outline our formulation of soliton equations based on the approach of Drinfel'd and Sokolov[BtK2, dGHM, DS, W].

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra, and $(\ ,\ )$ be the normalized invariant scalar product of $\mathfrak{g}$. The affine Lie algebra $\widehat{\mathfrak{g}}$ associated to $(\mathfrak{g}, (\ ,\ ))$ can be realized as

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

with the relations,

$$\begin{aligned}
&[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X, Y)K,\\
&[K, \widehat{\mathfrak{g}}] = 0, \quad [d, X \otimes z^n] = nX \otimes z^n,
\end{aligned}$$

for $X, Y \in \widehat{\mathfrak{g}}$, $m, n \in \mathbb{Z}$ [K].

To construct integrable hierarchies, Heisenberg subalgebras of $\widehat{\mathfrak{g}}$ play a crucial role. It is known that non-equivalent Heisenberg subalgebras are classified by conjugacy classes of the Weyl group of $\mathfrak{g}$ [KP, dGHM]. We denote by $\mathcal{H}^{[w]}$ the Heisenberg subalgebra associated with the conjugacy class $[w]$:

$$\mathcal{H}^{[w]} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\Lambda_n^{[w]} \oplus \mathbb{C}K.$$

Once we fix a basis of Heisenberg subalgebra $\{\Lambda_n^{[w]}\}_{n\in\mathbb{Z}}$, there is an associated gradation d_w that is natural on $\{\Lambda_n^{[w]}\}_{n\in\mathbb{Z}}$:

$$[d_w, \Lambda_n^{[w]}] = n\Lambda_n^{[w]}.$$

The gradation d_w induces a $\mathbb{Z}$-grading on $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}} \widehat{\mathfrak{g}}_j^{[w]}, \quad \widehat{\mathfrak{g}}_j^{[w]} = \{x \in \widehat{\mathfrak{g}}\ ;\ [d_w, x] = jx\}.$$

For an integer k, we use the notation

$$\widehat{\mathfrak{g}}_{\geq k}^{[w]} = \bigoplus_{j\geq k} \widehat{\mathfrak{g}}_j^{[w]}, \quad \widehat{\mathfrak{g}}_{<k}^{[w]} = \bigoplus_{j<k} \widehat{\mathfrak{g}}_j^{[w]}.$$

We now consider a Kac-Moody group $\widehat{G}$ formed by exponentiating the action of $\widehat{\mathfrak{g}}$ on a integrable module. Throughout this paper, we assume that the exponentiated action of an element of the positive degree subalgebra of $\mathcal{H}^{[w]}$ is well-defined. We remark that all of the representations used in what follows belong to this category. We denote by $\widehat{G}_{\geq 0}^{[w]}$ and $\widehat{G}_{<0}^{[w]}$ the subgroups correspond to the subalgebras $\widehat{\mathfrak{g}}_{\geq 0}^{[w]}$ and $\widehat{\mathfrak{g}}_{<0}^{[w]}$, respectively.

Starting from $g(0) \in \widehat{G}$, we define time-evolutions with time variable $t = (t_1, t_2, \dots)$ using the Heisenberg subalgebra $\{\Lambda_n^{[w']}\}_{n\in\mathbb{Z}}$ associated with a conjugacy class $[w']$:

$$g(t) \stackrel{\text{def}}{=} \exp\left(\sum_{n>0} t_n \Lambda_n^{[w']}\right) g(0), \tag{2.1}$$

which satisfies the following differential equation,

$$\frac{\partial g(t)}{\partial t_n} = \Lambda_n^{[w']} g(t), \quad n = 1, 2, \dots \tag{2.2}$$

In what follows, we shall assume the existence and the uniqueness of the Gauss decomposition with respect to the gradation d_w:

$$g(t) = \{g_{<0}^{[w]}(t)\}^{-1} g_{\geq 0}^{[w]}(t), \qquad g_{<0}^{[w]}(t) \in \widehat{G}_{<0}^{[w]}, \quad g_{\geq 0}^{[w]}(t) \in \widehat{G}_{\geq 0}^{[w]}. \tag{2.3}$$

A detailed discussion about this assumption is in [BtK, W] for instance. Note that the conjugacy classes of Weyl group $[w]$ of (2.3) and $[w']$ of (2.1) is not necessary equal.

From (2.2) and (2.3), we have

$$\frac{\partial g_{<0}^{[w]}}{\partial t_n} = B_n g_{<0}^{[w]} - g_{<0}^{[w]} \Lambda_n^{[w']}, \tag{2.4}$$

$$\frac{\partial g_{\geq 0}^{[w]}}{\partial t_n} = B_n g_{\geq 0}^{[w]}, \tag{2.5}$$

where $B_n = B_n(t)$ is defined by

$$B_n(t) \stackrel{\text{def}}{=} \left(g_{<0}^{[w]}(t) \Lambda_n^{[w']} g_{<0}^{[w]}(t)^{-1} \right)_{\geq 0}^{[w]} \in \widehat{\mathfrak{g}}_{\geq 0}^{[w]}. \tag{2.6}$$

We call (2.4) and (2.5) the Sato-Wilson equations. The compatibility conditions for (2.4) or (2.5) give rise to the zero-curvature (or Zakharov-Shabat) equations,

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \quad m, n = 1, 2, \ldots, \tag{2.7}$$

which gives a hierarchy of soliton equations.

Note that de Groot et al. imposed the condition $w' = w$ and in the definition (2.6) of B_n, they used a projection with respect to d_w or a less gradation than d_w for an order of gradations [dGHM]. However the formulas (2.4) and (2.5) is valid without the relation for w and w'. In this sense, our formulation can be regarded as an extension of the generalized Drinfel'd-Sokolov hierarchy.

2.2 Hierarchy of the derivative NLS equation

Hereafter we consider only the $\widehat{\mathfrak{sl}}_2$-case to treat the ∂NLS hierarchy. The generators of $\mathfrak{sl}_2$ is denoted by E, F and H as usual:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

We will use the abbreviation $X_n = X \otimes z^n$ for $X = E, F, H$.

In the case of $\mathfrak{sl}_2$, corresponding Weyl group is the symmetric group $\mathfrak{S}_2$ of order 2, generated by the simple transposition σ. The gradation corresponds to Id$\in \mathfrak{S}_2$ is given by the element d, and called "homogeneous". The gradation corresponds to $\sigma \in \mathfrak{S}_2$ is called "principal", given by $d_{\mathrm{p}} = 2d + \frac{1}{2} H_0$.

We choose the Heisenberg subalgebra of homogeneous-type,

$$\Lambda_n^{[\mathrm{h}]} \stackrel{\text{def}}{=} H_n, \tag{2.8}$$

and the triangular decomposition of principal-type,

$$\widehat{\mathfrak{sl}}_2 = \left(\widehat{\mathfrak{sl}}_2\right)_{<0}^{[\mathrm{p}]} \oplus \left(\widehat{\mathfrak{sl}}_2\right)_{\geq 0}^{[\mathrm{p}]}.$$

In other words, we have chosen $w' = \mathrm{Id}$ in (2.1) and $w = \sigma$ in (2.3). We stress that this choice does not fit to the condition $w' \geq w$ in the sense of the Bruhat order, and

thus does not fall into the category treated in [dGHM]. Note that the homogeneous Heisenberg subalgebra (2.8) has even principal grades:

$$[d_{\mathrm{p}}, \Lambda_n^{[\mathrm{h}]}] = 2n\Lambda_n^{[\mathrm{h}]}.$$

We consider a formal series expansion of $\log g_{<0}^{[\mathrm{p}]}(t) \in \widehat{\mathfrak{sl}}_2$ as follows:

$$\begin{aligned}\log g_{<0}^{[\mathrm{p}]}(t) = &\{q(t)E_{-1} + r(t)F_0\} + u(t)H_{-1} \\ &+ \{v_1(t)E_{-2} + v_2(t)F_{-1}\} + w(t)H_{-2} + \cdots .\end{aligned} \tag{2.9}$$

By straightforward calculations, we can obtain the expression for B_1:

$$B_1 = H_1 + (-2qE_0 + 2rF_1) + \{2qrH_0 - (qr + 2u)K\}. \tag{2.10}$$

Lemma 1.

$$\frac{\partial q}{\partial t_1} = 2v_1 + 2qu + \frac{4}{3}q^2 r, \quad \frac{\partial r}{\partial t_1} = -2v_2 + 2ru - \frac{4}{3}qr^2 \tag{2.11}$$

Proof. In the present case, the Sato-Wilson equation (2.4) with $n = 1$ is equivalent to the following equation in $\widehat{\mathfrak{sl}}_2$:

$$\frac{\partial g_{<0}^{[\mathrm{p}]}}{\partial t_1} (g_{<0}^{[\mathrm{p}]})^{-1} = B_1 - g_{<0}^{[\mathrm{p}]} H_1 (g_{<0}^{[\mathrm{p}]})^{-1}. \tag{2.12}$$

Comparing the $(\cdot)_{-1}^{[\mathrm{p}]}$-part of the both side of (2.12), we can derive the desirous result. □

This lemma gives us the expression for B_2:

$$\begin{aligned}B_2 = &H_2 + (-2qE_1 + 2rF_2) + 2qrH_1 + (-q'E_0 - r'F_1) \\ &+ \left\{ \left(q'r - qr' - 2q^2r^2\right) H_0 + \left(-4w - rv_1 - \frac{2}{3}qru - \frac{1}{3}q^2r^2\right) K \right\}.\end{aligned} \tag{2.13}$$

Here and throughout this paper, $'$ denotes partial differentiation with respect to t_1. Substituting these expressions into (2.7), we can obtain the ∂NLS equation (1.3) for $x = t_1, t = t_2$. In this sense, the hierarchy now we consider is nothing but the ∂NLS hierarchy.

A level-0 realization of $\widehat{\mathfrak{sl}}_2$ is given by

$$\begin{aligned}&E_n \mapsto \begin{pmatrix} 0 & z^n \\ 0 & 0 \end{pmatrix}, \quad F_n \mapsto \begin{pmatrix} 0 & 0 \\ z^n & 0 \end{pmatrix}, \quad H_n \mapsto \begin{pmatrix} z^n & 0 \\ 0 & -z^n \end{pmatrix}, \\ &K \mapsto 0, \quad d \mapsto z\frac{d}{dz}.\end{aligned} \tag{2.14}$$

Using this realization, we can express B_1 and B_2 as 2×2 matrices:

$$\begin{aligned}\boldsymbol{B}_1 &= \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} + \begin{pmatrix} 0 & -2q \\ 2zr & 0 \end{pmatrix} + \begin{pmatrix} 2qr & 0 \\ 0 & -2qr \end{pmatrix}, \\ \boldsymbol{B}_2 &= \begin{pmatrix} z^2 & 0 \\ 0 & -z^2 \end{pmatrix} + \begin{pmatrix} 0 & -2zq \\ 2z^2r & 0 \end{pmatrix} + \begin{pmatrix} 2zqr & 0 \\ 0 & -2zqr \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & -q' \\ -zr' & 0 \end{pmatrix} + \begin{pmatrix} q'r - qr' - 2q^2r^2 & 0 \\ 0 & qr' - q'r + 2q^2r^2 \end{pmatrix}.\end{aligned}$$

These matrices give a Lax pair for the ∂NLS equation but are different from the conventional one (cf. [WS]). To reproduce the conventional Lax pair, we use the other level-0 realization of $\widehat{\mathfrak{sl}}_2$ given by

$$
\begin{aligned}
&E_n \mapsto \begin{pmatrix} 0 & \lambda^{2n+1} \\ 0 & 0 \end{pmatrix}, \quad F_n \mapsto \begin{pmatrix} 0 & 0 \\ \lambda^{2n-1} & 0 \end{pmatrix}, \quad H_n \mapsto \begin{pmatrix} \lambda^{2n} & 0 \\ 0 & -\lambda^{2n} \end{pmatrix}, \\
&K \mapsto 0, \quad d \mapsto \frac{1}{2}\left\{ z\frac{d}{dz} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.
\end{aligned} \tag{2.15}
$$

From this realization, we obtain

$$
\begin{aligned}
\boldsymbol{B}_1 &= \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & -2q \\ 2r & 0 \end{pmatrix} + \begin{pmatrix} 2qr & 0 \\ 0 & -2qr \end{pmatrix}, \\
\boldsymbol{B}_2 &= \lambda^4 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda^3 \begin{pmatrix} 0 & -2q \\ 2r & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 2qr & 0 \\ 0 & -2qr \end{pmatrix} \\
&\quad + \lambda \begin{pmatrix} 0 & -q' \\ -r' & 0 \end{pmatrix} + \begin{pmatrix} q'r - qr' - 2q^2r^2 & 0 \\ 0 & qr' - q'r + 2q^2r^2 \end{pmatrix}.
\end{aligned}
$$

For the latter use, we decompose $g^{[\mathrm{p}]}_{\geq 0}(t)$ into grade 0 and > 0 part:

$$
g^{[\mathrm{p}]}_{\geq 0}(t) = g^{[\mathrm{p}]}_{0}(t) g^{[\mathrm{p}]}_{>0}(t). \tag{2.16}
$$

Substituting (2.16) into (2.5), we obtain

$$
\begin{aligned}
\frac{\partial g^{[\mathrm{p}]}_0}{\partial t_n} &= \left((g^{[\mathrm{p}]}_0)^{-1} B_n g^{[\mathrm{p}]}_0 \right)^{[\mathrm{p}]}_0 g^{[\mathrm{p}]}_0, \\
\frac{\partial g^{[\mathrm{p}]}_{>0}}{\partial t_n} &= \left\{ (g^{[\mathrm{p}]}_0)^{-1} B_n g^{[\mathrm{p}]}_0 - \left((g^{[\mathrm{p}]}_0)^{-1} B_n g^{[\mathrm{p}]}_0 \right)^{[\mathrm{p}]}_0 \right\} g^{[\mathrm{p}]}_{>0}.
\end{aligned}
$$

From these differential equations together with (2.10), (2.13) and formal expansions,

$$
\log g^{[\mathrm{p}]}_0(t) = \phi(t) H_0 + \psi(t) K, \tag{2.17}
$$

$$
\log g^{[\mathrm{p}]}_{>0}(t) = a(t) E_0 + b(t) F_1 + c(t) H_1 + \cdots, \tag{2.18}
$$

it follows that the functions $\phi(t)$, $a(t)$, $b(t)$ satisfy the equations,

$$
\frac{\partial \phi}{\partial t_1} = 2qr, \qquad \frac{\partial \phi}{\partial t_2} = q'r - qr' - 2q^2r^2, \tag{2.19}
$$

$$
\frac{\partial a}{\partial t_1} = -2q\, \mathrm{e}^{-2\phi}, \qquad \frac{\partial a}{\partial t_2} = -q'\, \mathrm{e}^{-2\phi}, \tag{2.20}
$$

$$
\frac{\partial b}{\partial t_1} = 2r\, \mathrm{e}^{2\phi}, \qquad \frac{\partial b}{\partial t_2} = -r'\, \mathrm{e}^{2\phi}. \tag{2.21}
$$

2.3 Miura-type transformation to the NLS equation

The homogeneous hierarchy that includes the NLS equation is obtained by taking $w' = \mathrm{Id}$ in the time-evolution (2.1), same as ∂NLS hierarchy, and $w = \mathrm{Id}$ in the Gauss decomposition (2.3). We denote the result of the decomposition by

$$g(t) = \{g^{[\mathrm{h}]}_{<0}(t)\}^{-1} g^{[\mathrm{h}]}_{\geq 0}(t), \qquad g^{[\mathrm{h}]}_{<0}(t) \in \widehat{G}^{[\mathrm{Id}]}_{<0}, \quad g^{[\mathrm{h}]}_{<0}(t) \in \widehat{G}^{[\mathrm{Id}]}_{\geq 0}.$$

The relation between this system and the ∂NLS hierarchy is established by the Miura-type transformation, which is an analog of the Miura transformation in the case of the KdV and the mKdV equations. For $g^{[\mathrm{p}]}_{<0}(t)$ of (2.9), we put

$$G \stackrel{\mathrm{def}}{=} \exp(-r(t)F_0)$$

and consider the decomposition,

$$g(t) = \{Gg^{[\mathrm{p}]}_{<0}(t)\}^{-1} Gg^{[\mathrm{p}]}_{\geq 0}, \qquad Gg^{[\mathrm{p}]}_{<0}(t) \in \widehat{G}^{[\mathrm{Id}]}_{<0}, \quad Gg^{[\mathrm{p}]}_{\geq 0}(t) \in \widehat{G}^{[\mathrm{Id}]}_{\geq 0}.$$

The assumption of the uniqueness of the Gauss decomposition causes

$$g^{[\mathrm{h}]}_{<0}(t) = Gg^{[\mathrm{p}]}_{<0}(t) = \exp(qE_{-1} + uH_{-1} + \cdots) \in \widehat{G}^{[\mathrm{Id}]}_{<0}, \tag{2.22}$$

$$g^{[\mathrm{h}]}_{\geq 0}(t) = Gg^{[\mathrm{p}]}_{\geq 0}(t) \in \widehat{G}^{[\mathrm{Id}]}_{\geq 0}. \tag{2.23}$$

These relations can be considered as Miura-type transformation in affine Lie group. By the equation (2.4), we can write

$$\frac{\partial}{\partial t_n} - B_n = g^{[\mathrm{p}]}_{<0} \left(\frac{\partial}{\partial t_n} - H_n \right) (g^{[\mathrm{p}]}_{<0})^{-1}. \tag{2.24}$$

Then we can describe the transformation in terms of B_n by translating $g^{[\mathrm{p}]}_{<0}(t)$ to $g^{[\mathrm{h}]}_{<0}(t)$ in (2.24) and we obtain

$$\frac{\partial}{\partial t_n} - \tilde{B}_n \stackrel{\mathrm{def}}{=} g^{[\mathrm{h}]}_{<0} \left(\frac{\partial}{\partial t_n} - H_n \right) (g^{[\mathrm{h}]}_{<0})^{-1} = G \left(\frac{\partial}{\partial t_n} - B_n \right) G^{-1}.$$

Note that this transformation preserves the zero-curvature equations (2.7). The relation between $\tilde{B}_n$ and B_n can be described as follows:

$$\tilde{B}_n = GB_nG^{-1} + \frac{\partial G}{\partial t_n} G^{-1}. \tag{2.25}$$

For $n = 1, 2$, we obtain

$$\tilde{B}_1 = H_1 + (-2qE_0 - (r' + 2qr^2)F_0) - (qr + 2u)K, \tag{2.26}$$

$$\begin{aligned} \tilde{B}_2 = & H_2 + (-2qE_1 - (r' + 2qr^2)F_1) - q(r' + 2qr^2)H_0 \\ & - q'E_0 + \left(\frac{r'}{2} + qr^2 \right)' F_0 + \left(-4w - rv_1 - \frac{2}{3}qru - \frac{1}{3}q^2r^2 \right) K. \end{aligned} \tag{2.27}$$

Here we have used the ∂NLS equation (1.3) to eliminate r_t in $\tilde{B}_2$.

If we put

$$\hat{r} = -\frac{r'}{2} - qr^2, \tag{2.28}$$

then the zero-curvature equation for $\tilde{B}_1$ and $\tilde{B}_2$ gives the NLS equation (1.4).

2.4 Gauge transformation to a generalized ∂NLS equation

There are several different kind of derivative NLS equations [ARS, CLL, GI, KSS, KN, K]. By extending the approach of [WS], Kundu obtained the generalized ∂NLS equation [K],

$$\begin{cases} Q_t = \frac{1}{2}Q'' + 2cQRQ' + 2(c-1)Q^2R' - 2(c-1)(c-2)Q^3R^2, \\ R_t = -\frac{1}{2}R'' + 2cQRR' + 2(c-1)R^2Q' + 2(c-1)(c-2)Q^2R^3. \end{cases} \tag{2.29}$$

Here c is a complex parameter. The equation (2.29) include the Kaup-Newell equation ($c=1$) [KN], the Chen-Lee-Liu equation ($c=2$) [CLL] and also (1.3) ($c=0$) as special cases. We can obtain the equation (2.29) by the gauge transformation of type (2.25) with respect to $g_0^{[p]}(t)^{-c/2} = \exp(-(c\phi/2)H_0)$:

$$\begin{aligned} \frac{\partial}{\partial t_n} - B_n \mapsto g_0^{[p]}(t)^{-c/2}\left(\frac{\partial}{\partial t_n} - B_n\right) g_0^{[p]}(t)^{c/2} \\ = \frac{\partial}{\partial t_n} - g_0^{[p]}(t)^{-c/2} B_n g_0^{[p]}(t)^{c/2} + g_0^{[p]}(t)^{-c/2}\frac{\partial g_0^{[p]}(t)^{c/2}}{\partial t_n} \end{aligned}$$

and put

$$C_n \stackrel{\text{def}}{=} g_0^{[p]}(t)^{-c/2} B_n g_0^{[p]}(t)^{c/2} - g_0^{[p]}(t)^{-c/2}\frac{\partial g_0^{[p]}(t)^{c/2}}{\partial t_n}.$$

Then for $n=1,2$, we have

$$\begin{aligned} C_1 =& H_1 + (-2qe^{-c\phi}E_0 + 2re^{c\phi}F_1) - (c-2)qrH_0 - (qr+2u)K \\ C_2 =& H_2 + (-2qe^{-c\phi}E_1 + 2re^{c\phi}F_2) + 2qrH_1 \\ &+ (-q'e^{-c\phi}E_0 - r'e^{c\phi}F_1) + (1-c/2)(q'r - qr' - 2q^2r^2)H_0 \\ &+ \left(-4w - rv_1 - \frac{2}{3}qru - \frac{1}{3}q^2r^2\right)K. \end{aligned}$$

Here we have used the relation (2.19). We introduce the new variables

$$Q(t) \stackrel{\text{def}}{=} qe^{-c\phi}, \qquad R(t) \stackrel{\text{def}}{=} re^{c\phi}.$$

By (2.19), the derivatives of these functions are written as

$$Q' = q'e^{-c\phi} - 2cQ^2R, \quad R' = r'e^{c\phi} + 2cR^2Q.$$

Then the zero-curvature equation for C_1 and C_2 result in the equations (2.29). Especially, the Lax operators C_1, C_2 for the Kaup-Newell equation and the Chen-Lee-Liu equation realized as matrix form (2.15) are identified with that of [WS].

3 Actions of affine Weyl group to the ∂NLS hierarchy

In this section, we discuss symmetries of the ∂NLS hierarchy in terms of the affine Weyl group $W(A_1^{(1)}) = \langle s_0, s_1\rangle$ with the relations $s_0^2 = s_1^2 = \mathrm{Id}$.

Let V be an integrable module of $\widehat{\mathfrak{sl}}_2$. The affine Weyl group $W(A_1^{(1)})$ acts on V as follows [K]:

$$s_j = \exp(f_j)\exp(-e_j)\exp(f_j) \quad (j = 0, 1), \tag{3.1}$$

where e_j, f_j are the Chevalley generators of $\widehat{\mathfrak{sl}}_2$ given by

$$\begin{aligned} e_0 &= F_1, & f_0 &= E_{-1}, & h_0 &= K - H_0, \\ e_1 &= E_0, & f_1 &= F_0, & h_1 &= H_0. \end{aligned}$$

Note that e_j, h_j, f_j $(j = 0, 1)$ have principal grades $1, 0, -1$, respectively. Under the level-0 realization (2.14), we can describe them as follows:

$$s_0 \mapsto \begin{pmatrix} 0 & z^{-1} \\ -z & 0 \end{pmatrix}, \quad s_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

The generators s_0, s_1 act naturally on $g(0)$ of (2.1) in two different ways.

3.1 Left-action

We consider the left-action of s_j $(j = 0, 1)$ of the form $s_j^{-1} g(0)$. Applying the principal Gauss decomposition (2.3) to $\exp[\sum_n t_n H_n]\, s_j^{-1} g(0)$, we define $s_j^{\mathrm{L}}(g_{<0}(t))$ and $s_j^{\mathrm{L}}(g_{\geq 0}(t))$ as

$$\left\{ s_j^{\mathrm{L}}(g_{<0}(t)) \right\}^{-1} s_j^{\mathrm{L}}(g_{\geq 0}(t)) = \exp\left[\sum_{n>0} t_n H_n \right] s_j^{-1} g(0). \tag{3.3}$$

This decomposition induces an action of s_j on the variables $q(t)$, $r(t)$.

Theorem 1. *Assume that the Gauss decomposition* (3.3) *exists uniquely. Then one can write down the action of s_j $(j = 0, 1)$ explicitly:*

$$s_0^{\mathrm{L}} : q(t) \mapsto -\frac{1}{q(-t)}, \quad r(t) \mapsto q(-t)^2 r(-t) - \frac{1}{2} q'(-t), \tag{3.4}$$

$$s_1^{\mathrm{L}} : q(t) \mapsto q(-t) r(-t)^2 + \frac{1}{2} r'(-t), \quad r(t) \mapsto -\frac{1}{r(-t)}. \tag{3.5}$$

Here $-t = (-t_1, -t_2, \dots)$.

Proof. Using the relation $s_j H_n s_j^{-1} = -H_n$ $(j = 0, 1,\ n = 1, 2, \dots)$, one can rewrite (3.3) as

$$\left\{ s_j^{\mathrm{L}}(g_{<0}(t)) \right\}^{-1} s_j^{\mathrm{L}}(g_{\geq 0}(t)) = s_j^{-1} g(-t) = \left\{ g_{<0}^{[\mathrm{p}]}(-t) s_j \right\}^{-1} g_{\geq 0}^{[\mathrm{p}]}(-t).$$

Next we consider the Gauss decomposition of $\left\{ g_{<0}^{[\mathrm{p}]}(-t) s_j \right\}^{-1}$:

$$\left\{ g_{<0}^{[\mathrm{p}]}(-t) s_j \right\}^{-1} = \left\{ \check{g}_{<0}^{(j)}(t) \right\}^{-1} \check{g}_{\geq 0}^{(j)}(t). \tag{3.6}$$

Assuming the uniqueness of the Gauss decomposition (3.3), one finds that

$$s_j^{\mathrm{L}}(g_{<0}(t)) = \check{g}_{<0}^{(j)}(t).$$

The decomposition (3.6) is equivalent to the condition,

$$g^{[\mathrm{p}]}_{<0}(-t)s_j\left\{\check{g}^{(j)}_{<0}(t)\right\}^{-1}\in\widehat{G}^{[\mathrm{p}]}_{\geq 0}. \tag{3.7}$$

We introduce a formal series expansion of $\log\check{g}^{(j)}_{<0}(t)$ as

$$\begin{aligned}\log\check{g}^{(j)}_{<0}(t)=&\{\check{q}_j(t)E_{-1}+\check{r}_j(t)F_0\}+\check{u}_j(t)H_{-1}\\&+\{\check{v}_{1,j}(t)E_{-2}+\check{v}_{2,j}(t)F_{-1}\}+\check{w}_j(t)H_{-2}+\cdots.\end{aligned} \tag{3.8}$$

Substituting $g^{[\mathrm{p}]}_{<0}(-t)$ (2.9), s_j (3.1) and (3.8) into $g^{[\mathrm{p}]}_{<0}(-t)s_j\left\{\check{g}^{(j)}_{<0}(t)\right\}^{-1}$ and using the realization (2.14), we can rewrite the condition (3.7) as relations between the coefficients of $g^{[\mathrm{p}]}_{<0}(-t)$ (2.9) and $\check{g}^{[\mathrm{p}]}_{<0}(t)$ (3.8). For example, in the case of s_1, we have

$$\begin{aligned}&1+r(-t)\check{r}_1(t)=0,\\&u(-t)+\check{u}_1(t)+\frac{q(-t)r(-t)+\check{q}_1(t)\check{r}_1(t)}{2}=0,\\&v_2(-t)+\frac{q(-t)r(-t)^2}{6}+\check{q}_1(t)+\frac{r(-t)\check{q}_1(t)\check{r}_1(t)}{2}+r(-t)\check{u}_1(t)=0.\end{aligned}$$

From these relations together with (2.11), we obtain (3.5). The s_0-action (3.4) can be obtained in a similar way. □

We remark that $-s^{\mathrm{L}}_1(q(-t))$ coincides with $\hat{r}$ of (2.28). Thus $q(t)$ and $\hat{r}(t)=-s^{\mathrm{L}}_1(q(-t))$ solve the NLS equation (1.4).

3.2 Right-action

Next we consider the right-action of s_j $(j=0,1)$ of the form $g(0)s_j$, which induces another action of s_j on the variables $q(t)$, $r(t)$ through the decomposition,

$$\left\{s^{\mathrm{R}}_j(g^{[\mathrm{p}]}_{<0}(t))\right\}^{-1}s^{\mathrm{R}}_j(g^{[\mathrm{p}]}_{\geq 0}(t))=g(t)s_j. \tag{3.9}$$

Theorem 2. *Assume that the Gauss decomposition* (3.9) *exists uniquely. Then one can write down the action of* s_j $(j=0,1)$ *explicitly:*

$$s^{\mathrm{R}}_0: q(t)\mapsto q(t)-\frac{1}{\tilde{\psi}_0(t)},\quad r(t)\mapsto r(t), \tag{3.10}$$

$$s^{\mathrm{R}}_1: q(t)\mapsto q(t),\quad r(t)\mapsto r(t)-\frac{1}{\tilde{\psi}_1(t)}. \tag{3.11}$$

Here $\tilde{\psi}_0(t)$ *and* $\tilde{\psi}_1(t)$ *satisfy the following differential equations,*

$$\frac{\partial\tilde{\psi}_0}{\partial t_1}=2r-4qr\tilde{\psi}_0,\qquad\frac{\partial\tilde{\psi}_0}{\partial t_2}=-r'-2(q'r-qr'-2q^2r^2)\tilde{\psi}_0, \tag{3.12}$$

$$\frac{\partial\tilde{\psi}_1}{\partial t_1}=-2q+4qr\tilde{\psi}_1,\qquad\frac{\partial\tilde{\psi}_1}{\partial t_2}=-q'+2(q'r-qr'-2q^2r^2)\tilde{\psi}_1. \tag{3.13}$$

Proof. We consider the Gauss decomposition of $g^{[\mathrm{p}]}_{\geq 0}(t)s_j$:

$$g^{[\mathrm{p}]}_{\geq 0}(t)s_j = \left\{\tilde{g}^{(j)}_{<0}(t)\right\}^{-1} \tilde{g}^{(j)}_{\geq 0}(t). \tag{3.14}$$

Assuming the uniqueness of the Gauss decomposition (3.9), one finds that

$$s^{\mathrm{R}}_j(g^{[\mathrm{p}]}_{<0}(t)) = \tilde{g}^{(j)}_{<0}(t)g^{[\mathrm{p}]}_{<0}(t). \tag{3.15}$$

The decomposition (3.14) is equivalent to

$$g^{[\mathrm{p}]}_{\geq 0}(t)s_j \left\{\tilde{g}^{(j)}_{\geq 0}(t)\right\}^{-1} = \left\{\tilde{g}^{(j)}_{<0}(t)\right\}^{-1} \in \widehat{G}^{[\mathrm{p}]}_{<0}. \tag{3.16}$$

In the realization (2.14), substituting formal expansions $g^{[\mathrm{p}]}_{\geq 0}(t) = g^{[\mathrm{p}]}_{0}(t)g^{[\mathrm{p}]}_{>0}(t)$ (2.17), (2.18), $\tilde{g}^{(j)}_{\geq 0}(t) = \tilde{g}^{(j)}_{0}(t)\tilde{g}^{(j)}_{>0}(t)$

$$\log \tilde{g}^{(j)}_0(t) = \tilde{\phi}_j(t)H_0, \quad \log \tilde{g}^{(j)}_{>0}(t) = \tilde{a}_j(t)E_0 + \tilde{b}_j(t)F_1 + \tilde{c}_j(t)H_1 + \cdots,$$

and s_j (3.1) to (3.16), we have

$$\begin{aligned} &b\mathrm{e}^{\tilde{\phi}_0-\phi} = 1, \quad b\tilde{b}_0 = -1, \quad \ldots \\ &a\mathrm{e}^{\phi-\tilde{\phi}_1} = 1, \quad a\tilde{a}_1 = -1, \quad \ldots \end{aligned}$$

and

$$\tilde{g}^{(0)}_{<0}(t) = \exp[-\mathrm{e}^{\phi+\tilde{\phi}_0}E_{-1}], \qquad \tilde{g}^{(1)}_{<0}(t) = \exp[-\mathrm{e}^{-\phi-\tilde{\phi}_1}F_0].$$

Therefore, we obtain $s^{\mathrm{R}}_j(g^{[\mathrm{p}]}_{<0}(t))$ $(j = 0, 1)$ from (3.15). If we define

$$\tilde{\psi}_0 = \mathrm{e}^{-\phi-\tilde{\phi}_0} = \mathrm{e}^{-2\phi}b, \qquad \tilde{\psi}_1 = \mathrm{e}^{\phi+\tilde{\phi}_1} = \mathrm{e}^{2\phi}a \tag{3.17}$$

we have the formulas (3.10), (3.11). The differential equations (3.12), (3.13) follow from (2.19), (2.20) and (2.21). □

We remark that the right action can be described by the gauge transformation of the differential operators:

$$\frac{\partial}{\partial t_n} - s_j(B_n) = \tilde{g}^{(j)}_{<0}\left(\frac{\partial}{\partial t_n} - B_n\right)(\tilde{g}^{(j)}_{<0})^{-1} \qquad (j = 0, 1).$$

This construction of the Weyl group action is essentially the same as that of Noumi and Yamada [NY1, N]. We will discuss this point in what follows (See Section 4.4).

3.3 Extended affine Weyl group

We denote by π the Dynkin diagram automorphism of $A^{(1)}_1$-type defined by

$$\pi x_i = x_{i+1}\pi \quad (i = 0, 1,\ x = e, f, h),$$

where the subscripts are understood as elements of $\mathbb{Z}/2\mathbb{Z}$. We extend the affine Weyl group $W(A_1^{(1)})$ by adding the element π that satisfies algebraic relations

$$s_0^2 = s_1^2 = \pi^2 = 1, \quad \pi s_0 = s_1 \pi, \quad \pi s_1 = s_0 \pi. \tag{3.18}$$

We denote the extended Weyl group by $\widetilde{W}(A_1^{(1)})$. In the level-0 realization of the Chevalley generators (2.14), the automorphism π are realized by the adjoint action of the matrix

$$\begin{pmatrix} 0 & z^{-1/2} \\ -z^{1/2} & 0 \end{pmatrix}. \tag{3.19}$$

As in the case of s_j, the action of π on $g(0)$ induces a transformation on solutions of the ∂NLS equation through the Gauss decomposition,

$$\begin{aligned} \exp\left[\sum_n t_n H_n\right] \pi^{-1} g(0) \pi &= \pi^{-1} g(-t) \pi \\ &= \left\{\pi^{-1} g_{<0}^{[\mathrm{p}]}(-t)\pi\right\}^{-1} \pi^{-1} g_{\geq 0}^{[\mathrm{p}]}(-t)\pi. \end{aligned}$$

It follows that

$$\pi : \begin{cases} q(t) \mapsto -r(-t), \quad r(t) \mapsto -q(-t), \\ \phi(t) \mapsto -\phi(-t), \quad a(t) \mapsto -b(-t), \quad b(t) \mapsto -a(-t). \end{cases}$$

The sets of transformations $\langle s_0^{\mathrm{L}}, s_1^{\mathrm{L}}, \pi\rangle$ and $\langle s_0^{\mathrm{R}}, s_1^{\mathrm{R}}, \pi\rangle$ satisfy the relation of the extended affine Weyl group (3.18) and thus we have obtained two different realizations of $\widetilde{W}(A_1^{(1)})$.

4 Similarity reduction and monodromy problem

In this section, we formulate a similarity condition of soliton equations in algebraic framework and consider the relation to monodromy problem of a linear ordinary differential system.

4.1 Similarity condition for soliton equation

First, we impose a constraint for the initial data $g(0) = g(z;0)$:

$$[d, g(z;0)] = \alpha H_0 g(z;0) + \beta g(z;0) H_0 + \gamma g(z;0) K. \tag{4.1}$$

Here α, β, γ are complex parameters. This relation leads to the following constraint for $g(t) = g(z;t)$ of (2.1):

$$[d, g(z;t)] = \left(\alpha H_0 + \sum_{n>0} n t_n \frac{\partial}{\partial t_n}\right) g(z;t) + \beta g(z;t) H_0 + \gamma g(z;t) K, \tag{4.2}$$

because the generators of the homogeneous Heisenberg subalgebra H_n satisfy the condition,

$$\exp\left(\sum_{n>0} t_n H_n\right) \cdot d \cdot \exp\left(\sum_{n>0} t_n H_n\right) = d - \sum_{n>0} n t_n H_n. \tag{4.3}$$

Note that $d = z\partial_z$ is the derivation for the homogeneous gradation. These conditions correspond to the similarity conditions for $g(z;0)$ and $g(z;t)$:

$$g(\lambda z;0) = \lambda^{\alpha H_0} g(z;0)\lambda^{\beta H_0},$$
$$g(\lambda z;t) = \lambda^{\alpha H_0} g(z;\tilde{t})\lambda^{\beta H_0}, \quad \tilde{t} \stackrel{\mathrm{def}}{=} (\lambda t_1, \lambda^2 t_2, \dots)$$

by taking the exponential with respect to λ of the operators of both hand side of (4.1), (4.2) respectively.

By applying the Gauss decomposition to $g(z;t)$ with respect to the principal gradation, we obtain a constraint for $g^{[\mathrm{p}]}_{<0}(z;t)$ and $g^{[\mathrm{p}]}_{\geq 0}(z;t)$ such as

$$[d, g^{[\mathrm{p}]}_{<0}(z;t)] = [\alpha H_0, g^{[\mathrm{p}]}_{<0}(z;t)] + \sum_{n>0} n t_n \frac{\partial g^{[\mathrm{p}]}_{<0}(z;t)}{\partial t_n}, \tag{4.4}$$

$$[d, g^{[\mathrm{p}]}_{\geq 0}(z;t)] = \left(\alpha H_0 + \sum_{n>0} n t_n B_n\right) g^{[\mathrm{p}]}_{\geq 0}(z;t) + \beta g^{[\mathrm{p}]}_{\geq 0}(z;t) H_0 + \gamma g^{[\mathrm{p}]}_{\geq 0}(z;t) K. \tag{4.5}$$

These conditions correspond to the similarity conditions:

$$g^{[\mathrm{p}]}_{<0}(\lambda z;t) = \lambda^{\alpha H_0} g^{[\mathrm{p}]}_{<0}(z;\tilde{t})\lambda^{-\alpha H_0}, \quad g^{[\mathrm{p}]}_{\geq 0}(\lambda z;t) = \lambda^{\alpha H_0} g^{[\mathrm{p}]}_{\geq 0}(z,\tilde{t})\lambda^{\beta H_0} \tag{4.6}$$

Especially, the first few components of $g^{[\mathrm{p}]}_{<0}(z;t)$ (2.9) and $g^{[\mathrm{p}]}_{\geq 0}(z;t)$ (2.17) satisfy the following conditions:

$$q(\tilde{t}) = \lambda^{-2\alpha-1} q(t), \quad r(\tilde{t}) = \lambda^{2\alpha} r(t), \quad \phi(\tilde{t}) = \left(\log \lambda^{-(\alpha+\beta)}\right)\phi(t), \tag{4.7}$$

$$a(\tilde{t}) = \lambda^{2\beta} a(t), \quad b(\tilde{t}) = \lambda^{-2\beta+1} b(t). \tag{4.8}$$

Proposition 1. *If we set*

$$M = \alpha H_0 + \sum_{n>0} n t_n B_n, \tag{4.9}$$

then M and B_n $(n = 1, 2, \dots)$ satisfy the zero-curvature equations:

$$\left[z\frac{d}{dz} - M, \frac{\partial}{\partial t_n} - B_n\right] = 0. \tag{4.10}$$

Proof. By the definition (4.9) of M and relations (2.4), (2.5), (4.4), (4.5), we can describe

$$\begin{aligned} z\frac{d}{dz} - M &= g^{[\mathrm{p}]}_{<0}\left(z\frac{d}{dz} - \alpha H_0 - \sum_{n>0} n t_n H_n\right)(g^{[\mathrm{p}]}_{<0})^{-1} \\ &= g^{[\mathrm{p}]}_{\geq 0}\left(z\frac{d}{dz} + \beta H_0\right)(g^{[\mathrm{p}]}_{\geq 0})^{-1}. \end{aligned}$$

Therefore, by multiplying $(g^{[\mathrm{p}]}_{<0})^{-1}$ from the left and $g^{[\mathrm{p}]}_{<0}$ from the right to the formula

$$\left[z\frac{d}{dz} - \alpha H_0 - \sum_{n>0} n t_n H_n, \frac{\partial}{\partial t_m} - H_m\right] = 0 \quad (m = 1, 2, \dots)$$

or

$$\left[z\frac{d}{dz} + \beta H_0, \frac{\partial}{\partial t_m} - H_m\right] = 0 \quad (m = 1, 2, \dots),$$

we have the equation (4.10). □

4.2 Monodromy problem and Painlevé IV

We now fix a positive integer $l > 0$ and restrict the operator for the time evolution to $\exp[\sum_{n=0}^{l} t_n H_n]$, or we put $t_{l+1} = t_{l+2} = \cdots = 0$ in (4.3). Then M of (4.9) becomes a element of affine Lie algebra. Under the realization (2.14), we get a system of linear differential equations for a 2×2 matrix $Y = Y(z; t_1, \ldots, t_l)$:

$$z\frac{\partial}{\partial z}Y = MY, \qquad \frac{\partial}{\partial t_n}Y = B_n Y \quad (n = 1, \ldots, l). \tag{4.11}$$

This linear problem defines a monodromy preserving deformation of linear ordinary differential system, with regular singularity at $z = 0$ and irregular singularity of rank l at $z = \infty$. We regard $t_1, t_2, \ldots, t_l$ as a deformation parameter at ∞, and α, β as monodromy data at ∞, 0 respectively.

Hereafter, we set $l = 2$ and put $t = t_2 = 1/2$. Then M of (4.9) for B_1 (2.10) and B_2 (2.13) can be written as

$$\begin{aligned} M =& H_2 + (-2qE_1 + 2rF_2) + (x + 2qr)H_1 \\ &+ (-(2xq + q')E_0 + (2xr - r')F_1) + (\alpha + k)H_0, \end{aligned} \tag{4.12}$$

where

$$k = 2xqr + q'r - qr' - 2q^2r^2 = \left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\phi(t). \tag{4.13}$$

The second equality is given by (2.19). We set

$$\varphi = 2qr, \quad \psi_1 = -2x - \frac{q'}{q}, \quad \psi_0 = 2x - \frac{r'}{r}. \tag{4.14}$$

The compatibility condition (4.10) for the linear system (4.11) for the restricted M of (4.12) and B_1 of (2.10) present the following system of differential equations:

$$\varphi' = -\varphi(\psi_1 + \psi_0), \tag{4.15}$$

$$\psi_1' = \psi_1(2\varphi + \psi_1 + 2x) - 4\beta, \tag{4.16}$$

$$\psi_0' = \psi_0(-2\varphi + \psi_0 - 2x) - 2(2\beta - 1). \tag{4.17}$$

In addition, the similarity condition for ϕ of (4.7) fix the value of k (4.13):

$$k = -\alpha - \beta. \tag{4.18}$$

So we have the relation

$$\psi_0 - \psi_1 - \varphi + \frac{2(\alpha + \beta)}{\varphi} = 2x. \tag{4.19}$$

Proposition 2. *Each of the quantities φ, ψ_1 and $-\psi_0$ solve the fourth Painlevé equation* (1.5) *with the following parameters:*

	ν_1	ν_2
φ	$\alpha - 3\beta + 1$	$-2(\alpha + \beta)^2$
ψ_1	$-2\alpha - 1$	$-8\beta^2$
$-\psi_0$	2α	$-2(2\beta - 1)^2$

Proof. Differentiating (4.15), we obtain

$$\varphi'' = -\varphi'(\psi_0 + \psi_1) - \varphi\left(2(\varphi + x)(\psi_1 - \psi_0) + \psi_1^2 + \psi_0^2 - 8\beta + 2\right).$$

Then, by using the relations (4.15) and (4.19), we have P_{IV} (1.5) for φ. For ψ_1, the relations (4.15), (4.16) and (4.19) give the equations,

$$\psi_1' = 2\varphi\psi_1 + \psi_1^2 + 2x\psi_1 - 4\beta, \tag{4.20}$$

$$(\varphi\psi_1)' = \frac{(\varphi\psi_1)^2}{\psi_1} - \varphi\psi_1\left(\frac{4\beta}{\psi_1} + \psi_1\right) + 2(\alpha + \beta)\psi_1. \tag{4.21}$$

Then, differentiating the first equation (4.20) and eliminating $\varphi\psi_1$ by using (4.20) and (4.21), we have the fourth Painlevé equation for ψ_1. The other case $-\psi_0$ can be treated in the similar way. □

Remark 1. Ablowitz et al. presented the fourth Painlevé equation as a similarity reduction of ∂NLS [ARS]. In our notation, their results correspond to the equation for φ. However, their result has only one parameter β, and corresponds to the special case $\alpha = -1/4$. We give the fourth Painlevé equation with full parameters.

Remark 2. Our system of linear equations (4.11) is not a special case of the generalized Painlevé systems given by Noumi and Yamada [NY1, N]. Their system is based on the similarity reduction of the principal hierarchy.

Jimbo and Miwa [JM2] showed that P_{IV} is obtained as a similarity reduction of the NLS equation. Their results correspond to the Gauss decomposition of homogeneous-type in our setting. Since $g^{[h]}_{<0}(z;t)$ (2.22) and $g^{[h]}_{\geq 0}(z;t)$ (2.23) satisfy the same similarity conditions as (4.6):

$$g^{[h]}_{<0}(\lambda z;t) = \lambda^{\alpha H_0} g^{[h]}_{<0}(z;\tilde{t})\lambda^{-\alpha H_0}, \quad g^{[h]}_{\geq 0}(\lambda z;t) = \lambda^{\alpha H_0} g^{[h]}_{\geq 0}(z,\tilde{t})\lambda^{\beta H_0},$$

so the solutions of the NLS equation (1.4) satisfy the conditions

$$q(\tilde{t}) = \lambda^{-2\alpha-1} q(t), \quad \hat{r}(\tilde{t}) = \lambda^{2\alpha-1}\hat{r}(t).$$

By the same discussion as above, in the level-0 realization (2.14) of $\tilde{B}_1$ (2.26) and $\tilde{B}_2$ (2.27), we have the linear problem,

$$\frac{\partial}{\partial z} Y = A(z)Y, \qquad \frac{\partial}{\partial x} Y = B(z)Y,$$

with

$$A(z) = z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} x & -2q \\ 2\hat{r} & -x \end{pmatrix} + z^{-1}\begin{pmatrix} \alpha + 2q\hat{r} & -2xq - q' \\ 2x\hat{r} - \hat{r}' & -(\alpha + 2q\hat{r}) \end{pmatrix},$$

$$B(z) = z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -2q \\ 2\hat{r} & 0 \end{pmatrix}.$$

These can be identified with the result of [JM2].

Furthermore, we use (4.14), (4.19) and (2.28) to show that

$$\varphi\psi_1 = 4q\hat{r} + 2(\alpha+\beta). \tag{4.22}$$

Applying the relation (4.22) to the compatibility condition

$$\left[\frac{\partial}{\partial z} - A(z), \frac{\partial}{\partial x} - B(z)\right] = 0,$$

we have (4.20), (4.21) and thus obtain the fourth Painlevé equation for ψ_1.

4.3 Relations to Hamiltonian system

In [O], Okamoto showed that the fourth Painlevé equation (1.5) is equivalent to the Hamilton system,

$$y' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial y},$$

with the polynomial Hamiltonian,

$$H = yp^2 - y^2p - 2xpy - 2\theta_0 p + 2\theta_\infty y.$$

This is represented as the following system of equations for y and p:

$$\begin{aligned} y' &= y(2p - y - 2x) - 2\theta_0, \\ p' &= p(2q - p + 2x) - 2\theta_\infty. \end{aligned} \tag{4.23}$$

Our system (4.15)–(4.17) can be identified with (4.23) in two different ways. Firstly, if we eliminate ψ_0 from (4.15) by using (4.19), we have

$$\begin{aligned} \varphi' &= \varphi(-2\psi_1 - \varphi - 2x) + 2(\alpha+\beta), \\ \psi_1' &= \psi_1(2\varphi + \psi_1 + 2x) - 4\beta, \end{aligned}$$

which are equivalent to (4.23) with

$$(y,p) = (\varphi, -\psi_1), \quad (\theta_0, \theta_\infty) = (-\alpha-\beta, 2\beta).$$

Secondly, if we eliminate ψ_1, we have (4.23) with

$$(y,p) = (-\psi_0, -\varphi), \quad (\theta_0, \theta_\infty) = (\alpha+\beta, 2\beta - 1).$$

4.4 Weyl group symmetry for the fourth Painlevé equation

To construct a Weyl group symmetry for the similarity solution of the ∂NLS hierarchy, we examine a similarity conditions for $s_j^{-1}g(0)$ and $g(0)s_j$ ($j = 0, 1$). We have

$$\begin{aligned} [d, s_i^{-1}g(0)] &= \left\{\left(-\alpha - \frac{1}{2}\right)H_0 + \frac{h_i}{2}\right\} s_i^{-1}g(0) + \beta s_i^{-1}g(0)H_0 + \gamma s_i^{-1}g(0)K, \\ [d, g(0)s_i] &= \alpha H_0 g(0)s_i + g(0)s_i\left\{\left(-\beta + \frac{1}{2}\right)H_0 - \frac{h_i}{2}\right\} + \gamma g(0)K \end{aligned}$$

by using the relations (4.1) and

$$[d, s_i] = \frac{1}{2}h_i s_i - \frac{1}{4}[H_0, s_i] \qquad (i = 0, 1).$$

Therfore, we have two types of Weyl group actions for the parameters α, β

$$\begin{cases} s_0^{\mathrm{L}} : \alpha \mapsto -\alpha - 1, \quad \beta \mapsto \beta, \quad \gamma \mapsto \gamma + \dfrac{1}{2}, \\ s_1^{\mathrm{L}} : \alpha \mapsto -\alpha, \quad \beta \mapsto \beta, \quad \gamma \mapsto \gamma, \end{cases}$$

$$\begin{cases} s_0^{\mathrm{R}} : \alpha \mapsto \alpha, \quad \beta \mapsto -\beta + 1, \quad \gamma \mapsto \gamma - \dfrac{1}{2}, \\ s_1^{\mathrm{R}} : \alpha \mapsto \alpha, \quad \beta \mapsto -\beta, \quad \gamma \mapsto \gamma. \end{cases}$$

Next we consider the right-action of the affine Weyl group under the similarity condition (4.1). Applying the relation (4.18) to (3.12) and (3.13), we have

$$\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\tilde{\psi}_0 = 2xr - r' + 2(\alpha+\beta)\tilde{\psi}_0,$$
$$\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\tilde{\psi}_1 = -2xq - q' - 2(\alpha+\beta)\tilde{\psi}_1.$$

On the other hand, left-hand-side of these equations can be written in

$$\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\tilde{\psi}_0 = (2\alpha+1)\tilde{\psi}_0, \qquad \left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\tilde{\psi}_1 = -2\alpha\tilde{\psi}_1$$

by using (3.17), (4.7) and (4.8). Then, under the similarity condition, $\tilde{\psi}_0$ and $\tilde{\psi}_1$ can be expressed as

$$\tilde{\psi}_0 = \frac{2xr - r'}{1-2\beta} = \frac{r\psi_0}{1-2\beta}, \qquad \tilde{\psi}_1 = \frac{-2xq - q'}{2\beta} = \frac{q\psi_1}{2\beta}. \tag{4.24}$$

We remark that in the realization (2.14), the right-action of the affine Weyl group is represented as a compatibility of the gauge transformation

$$s_0 Y = \left(1 - \frac{1-2\beta}{r\psi_0}E_{-1}\right)Y, \qquad s_1 Y = \left(1 - \frac{2\beta}{q\psi_1}F_0\right)Y$$

for the linear system (4.11). This transformation is the same as the Weyl group symmetry of Painlevé type equation given by Noumi and Yamada [N].

In the level-0 realization (2.14), the action of the extended affine Weyl group can be obtained in the same manner. The matrix (3.19) satisfies the condition

$$[d, \pi] = -\frac{1}{4}[H_0, \pi],$$

and then the relation

$$[d, \pi^{-1}g(0)\pi] = \left\{\left(-\alpha - \frac{1}{2}\right)H_0\right\}\pi^{-1}g(0)\pi + \pi^{-1}g(0)\pi\left\{\left(-\beta + \frac{1}{2}\right)H_0\right\}$$

holds. Therefore the action of π for the parameters is given by

$$\pi : \alpha \mapsto -\alpha - \frac{1}{2}, \quad \beta \mapsto -\beta + \frac{1}{2}, \quad \gamma \mapsto \gamma.$$

4.5 Schlesinger transformations and discrete Painlevé equations

In the level-0 realization (2.14), we consider the local solutions of the linear system (4.11) at $z = \infty$ and $z = 0$. They are obtained by the following formal serieses:

$$\begin{aligned} Y^{(\infty)}(z;t) &= g^{[\mathrm{p}]}_{<0}(z;t) \exp\left((-\alpha \log z^{-1}) H_0 + \sum_{n=1}^{l} t_n H_n \right), \\ Y^{(0)}(z;t) &= g^{[\mathrm{p}]}_{\geq 0}(z;t) \exp\big((-\beta \log z) H_0 \big). \end{aligned}$$

Here $g^{[\mathrm{p}]}_{<0}(z;t)$ and $g^{[\mathrm{p}]}_{\geq 0}(z;t)$ are the solutions of ∂NLS hierarchy with the similarity conditions (4.4) (4.5) and the set of parameters $(-\alpha, -\beta)$ corresponds to the monodromy exponents. The Schlesinger transformation relates the two solutions Y and Y' of the isomonodromy problem for the equation at hand corresponding to different sets of parameters. The change in parameters $(-\alpha, -\beta)$ are integers or half-integers.

In the case of Painlevé IV, the Schlesinger transformation can be understood in terms of the extended affine Weyl group. If we consider the transformation $s_1^{\mathrm{R}} \pi s_1^{\mathrm{L}}$, the parameters $(-\alpha, -\beta)$ are transformed as

$$s_1^{\mathrm{R}} \pi s_1^{\mathrm{L}} : \ (-\alpha, -\beta) \mapsto \left(-\alpha + \frac{1}{2}, -\beta + \frac{1}{2} \right) \tag{4.25}$$

which corresponds to the Schlesinger transformation. Applying the realization (3.2), (3.19) of the extended affine Weyl group, we can describe this transformation as the compatibility condition of (4.11) with

$$M = \begin{pmatrix} z^2 + (x+\varphi)z - \beta & -2qz + q\psi_1 \\ 2rz^2 + r\psi_0 z & -z^2 - (x+\varphi)z + \beta \end{pmatrix}$$

and

$$\overline{Y} = RY, \qquad R = \begin{pmatrix} 0 & 0 \\ -r & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 0 & 1/r \\ 0 & r/\tilde{\psi}_0 \end{pmatrix} z^{-1/2}.$$

Note that $\tilde{\psi}_0$ is defined in (3.17). So the transformation of M is given by

$$\overline{M} = RMR^{-1} + z \frac{\partial R}{\partial z} R^{-1}.$$

Note that by the composition of left-action and π, the sign of the variable x does not change. Using the relations (4.19) and (4.24), we obtain the image of (φ, ψ_1) in terms of (φ, ψ_1):

$$\begin{aligned} \overline{\varphi} &= -2x - \varphi + \psi_0 + \frac{2(1-\beta)}{\psi_0}, \\ \overline{\psi_1} &= -\psi_0. \end{aligned}$$

We remark that ψ_0 can be written in (φ, ψ_1) by (4.19). Putting $\varphi = -2\chi_n$, $\psi_1 = -2\omega_n$ (and $\overline{\varphi} = -2\chi_{n+1}, \overline{\psi_1} = -2\omega_{n+1}$) we find

$$\chi_n + \chi_{n-1} = x - \omega_n + \frac{\beta}{\omega_n},$$
$$\omega_n + \omega_{n+1} = x - \chi_n + \frac{\alpha+\beta}{2\chi_n}.$$

These equations are reduced to the discrete Painlevé I equation (1.6) by putting $X_{2n} = \omega_n$, $X_{2n-1} = \chi_n$ for $n \in \mathbb{N}$.

Note that the Schlesinger transformation to another direction represented by $s_1^{\mathrm{R}} s_1^{\mathrm{L}} \pi$,

$$s_1^{\mathrm{R}} s_1^{\mathrm{L}} \pi : \ (-\alpha, -\beta) \mapsto \left(-\alpha - \frac{1}{2}, -\beta + \frac{1}{2}\right)$$

also gives the discrete Painlevé I equation for ψ_1 and $\varphi - 4\beta/\psi_1$.

5 Tau-functions and special solutions

In this section, we consider the basic representations of $\widehat{\mathfrak{sl}}_2$ [FK] to introduce "τ-functions". Let $|\varpi_j\rangle$ be a highest weight vector associated with the highest weight ϖ_j $(j = 0, 1)$, i.e.,

$$\begin{array}{l} e_i|\varpi_j\rangle = 0, \quad h_i|\varpi_j\rangle = \delta_{ij}|\varpi_j\rangle \quad (i, j = 0, 1), \\ f_0|\varpi_1\rangle = f_1|\varpi_0\rangle = 0. \end{array}$$

We denote by $L(\varpi_j)$ the basic representations with the highest weight ϖ_j, and by $L(\varpi_j)^*$ its dual space.

First we construct a realization of $L(\varpi_0) \oplus L(\varpi_1)$ on the space

$$V = \mathbb{C}[x_1, x_2, \ldots] \otimes \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}e^{n\alpha/2} \right),$$

where $\alpha \in (\mathbb{C}h_0 \oplus \mathbb{C}h_1)^*$ satisfies $\alpha(h_0) = -2$, $\alpha(h_1) = 2$. The representation (ρ, V) is given as follows:

$$\begin{aligned} \rho(H_j)\left(P(x) \otimes e^{n\alpha}\right) &= \begin{cases} 2\frac{\partial P(x)}{\partial x_j} \otimes e^{n\alpha} & (j \geq 1), \\ 2nP(x) \otimes e^{n\alpha} & (j = 0), \\ -jt_{-j}P(x) \otimes e^{n\alpha} & (j \leq -1), \end{cases} \\ \rho(K)\left(P(x) \otimes e^{n\alpha}\right) &= P(x) \otimes e^{n\alpha}, \\ \rho(d)\left(P(x) \otimes e^{n\alpha}\right) &= -\sum_{m=1}^{\infty} m x_m \frac{\partial P(x)}{\partial x_m} \otimes e^{n\alpha}. \end{aligned}$$

To describe the action of E_n and F_n, we introduce the generating series,

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1} \quad (X = E, F).$$

Then the action of $E(z)$ and $F(z)$ is given by the following operators ("vertex operators"):

$$\rho(E(z)) = e^{\eta(x,z)}e^{-2\eta(\tilde{\partial}_x,z^{-1})} \otimes e^{\alpha} z^{H_0},$$
$$\rho(F(z)) = e^{-\eta(x,z)}e^{2\eta(\tilde{\partial}_x,z^{-1})} \otimes e^{-\alpha} z^{-H_0},$$

with

$$\eta(x,z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} x_n z^n, \quad \eta(\tilde{\partial}_x, z^{-1}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial x_n}.$$

The representation (ρ, V) is decomposed as follows:

$$V = V_0 \oplus V_1, \quad V_j = \mathbb{C}[x_1, x_2, \ldots] \otimes \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}e^{(n+j/2)\alpha} \right) \quad (j = 0,1).$$

It is shown that each of the representation V_j $(j = 0, 1)$ is isomorphic to $L(\varpi_j)$ [FK], where the highest weight vector is given by $|\varpi_j\rangle = 1 \otimes e^{j\alpha/2}$.

Next we prepare several results on symmetric polynomials [M]. We denote by $p_j(z)$ the j-th elementary power-sum symmetric polynomial with respect to variables $z_1,\ldots,$ z_n:

$$p_j(z) = z_1^j + z_2^j + \cdots + z_n^j.$$

The Schur polynomial $S_\lambda(x)$, labeled by the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is expressed by

$$S_\lambda(x) = \det \left[s_{\lambda_i - i + j}(x) \right]_{1\le i,j\le n},$$

where $s_n(x)$ is the n-th elementary Schur polynomial defined by

$$\exp[\eta(x,\lambda)] = \sum_{j=0}^{\infty} s_j(x)\lambda^j,$$

and $s_n(x) = 0$ if $n < 0$.

We then introduce a scalar product in $\mathbb{C}[x_1, \ldots, x_n]$:

$$\langle P(x), Q(x)\rangle = \frac{1}{n}\,\text{C.T.}\left[P(x_j = \frac{p_j(z)}{n}) Q(x_j = \frac{p_{-j}(z)}{n}) \Delta(z)\Delta(z^{-1}) \right], \tag{5.1}$$

where C.T.$[f(z)]$ denotes the constant term of $f(z) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ and $\Delta(z) = \prod_{1\le i<j\le n}(z_i - z_j)$, $\Delta(z^{-1}) = \prod_{1\le i<j\le n}(z_i^{-1} - z_j^{-1})$. It is well-known that the Schur polynomials $\{S_\lambda\}$, associated with partitions $\{\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots), \lambda_j \in \mathbb{Z}_{\ge 0}\}$, are pairwise orthogonal with respect to the scalar product (5.1). The scalar product (5.1) induces a scalar product on $V = V_0 \oplus V_1$:

$$\langle P(x) \otimes e^{a\alpha}, Q(x) \otimes e^{b\alpha}\rangle = \delta_{ab}\langle P(x), Q(x)\rangle,$$

where $P(x), Q(x) \in \mathbb{C}[x]$ and $a, b \in \mathbb{Z}/2$. Since the Schur polynomials forms an orthogonal basis of $\mathbb{C}[x]$, an orthogonal basis of $V_0 \oplus V_1$ is given by $\{S_\lambda(x) \otimes e^{a\alpha}\}_{\lambda,a}$.

Following [BtK, HM], we define τ-functions associated with $g(t)$ of (2.1) as

$$\begin{aligned}\tau_n^{(j)}(t) &= \langle 1 \otimes e^{(n+j/2)\alpha}, g(t)(1 \otimes e^{j\alpha/2})\rangle \\ &= \langle 1 \otimes e^{(n+j/2)\alpha}, \{g_{<0}^{[\mathrm{p}]}(t)\}^{-1} g_0^{[\mathrm{p}]}(t)(1 \otimes e^{j\alpha/2})\rangle.\end{aligned}$$

We can express $q(t)$, $r(t)$ of (2.9) in terms of the τ-functions:

$$q(t) = -\frac{\tau_1^{(0)}(t)}{\tau_0^{(0)}(t)}, \quad r(t) = -\frac{\tau_{-1}^{(1)}(t)}{\tau_0^{(1)}(t)}.$$

As an example of concrete solutions, we construct polynomial-type τ-functions, which are written in terms of the Schur polynomials. To this aim, we prepare the two lemmas:

Lemma 2. *Let n be an integer. We have*

$$\rho(s_0 s_1)^n (1 \otimes e^{j\alpha/2}) = \epsilon_n (1 \otimes e^{(n+j/2)\alpha}),$$

where $j = 0, 1$ and $\epsilon_n = 1$ or -1 depending on the value of n.

Proof. This is a direct consequence of Lemma 12.6 of [K]. $\square$

Lemma 3. *(cf. [IY]) Let k be a non-negative half-integer. We have the following expression of weight vectors:*

$$\rho(e^{F_m})(1 \otimes e^{k\alpha}) = \sum_{n=0}^{2k-m} (-1)^{\frac{n(n+2m+1)}{2}} S_{\square(n,2k-m-n)}(t) \otimes e^{(k-n)\alpha},$$
$$\rho(e^{E_m})(1 \otimes e^{-k\alpha}) = \sum_{n=0}^{2k-m} (-1)^{\frac{n(n-1)}{2}} S_{\square(2k-m-n,n)}(t) \otimes e^{(n-k)\alpha},$$

where the rectangular Young diagram (k^n) is denoted by $\square(n,k)$.

Proof. A proof can be given by same way as Theorem 1 of [IY]. We omit the detail. $\square$

Now we define $g_l(0)$ as

$$g_l(0) = \begin{cases} e^{f_1}(s_0 s_1)^l & (l \geq 0), \\ e^{f_0}(s_0 s_1)^l & (l < 0), \end{cases} \tag{5.2}$$

for an integer l. Using Lemmas 2, 3, we can calculate the corresponding τ-functions explicitly:

$$\tau_n^{(0)} = \begin{cases} \epsilon_l (-1)^{\frac{(l-n)(l-n+1)}{2}} S_{\square(l-n,l+n)}(t) & (l \geq 0), \\ \epsilon_l (-1)^{\frac{(n-l)(n-l-1)}{2}} S_{\square(1-l-n,n-l)}(t) & (l < 0), \end{cases}$$
$$\tau_n^{(1)} = \begin{cases} \epsilon_l (-1)^{\frac{(l-n)(l-n+1)}{2}} S_{\square(l-n,l+n+1)}(t) & (l \geq 0), \\ \epsilon_l (-1)^{\frac{(n-l)(n-l-1)}{2}} S_{\square(-l-n,n-l)}(t) & (l < 0). \end{cases}$$

These τ-functions give rational solutions of the ∂NLS equation (1.3).

Furthermore, straightforward calculations show that $g_l(0)$ of (5.2) satisfies the reduction condition (4.1) with the following parameters:

$$l \geq 0: \ \alpha = 0, \quad \beta = -l,$$
$$l < 0: \ \alpha = -\frac{1}{2}, \quad \beta = \frac{1}{2} - l.$$

Hence we can perform the similarity reduction to the rational solutions given above and obtain rational solutions for the Painlevé IV. In this case, the Schur polynomials $p_n(t)$ are degenerated to the Hermite polynomials $H_n(t)$:

$$\begin{aligned}\exp(zt_1 + z^2 t_2 + \cdots)\big|_{t_1=x,\, t_2=1/2,\, t_3=t_4=\cdots=0} \\ = \exp(xz + z^2/2) = \sum_{n\in\mathbb{Z}} H_n(t) z^n.\end{aligned}$$

If we introduce discrete time evolution as (4.25),

$$g_l(0;n) = s_1^{\mathrm{R}} \pi s_1^{\mathrm{L}}(g_l(0)) = s_0^{-1}\pi^{-1}(g_l(0))\pi s_1,$$

the corresponding rational solutions solve the discrete Painlevé equation (1.6) as discussed in the section 4.5. We remark that the rational solutions for the discrete Painlevé I (1.6) constructed in [OKS] are essentially the same as the above.

6 Concluding remarks

We have formulated the hierarchy of the ∂NLS equation and introduced a systematic method for similarity reductions to Painlevé-type equations. We used the fermionic representation of $\widehat{\mathfrak{sl}}_2$ to construct rational solutions. We remark that the rational solutions can be expressed as ratio of Wronski-type determinants, which is discussed in [KK].

As pointed out by Okamoto [O], the fourth Painlevé equation has the Weyl group symmetry of $\widetilde{W}(A_2^{(1)})$-type. Adler [A] and Noumi and Yamada [NY2] propose a new representation of $\mathrm{P_{IV}}$, in which the $\widetilde{W}(A_2^{(1)})$ symmetries become clearly visible. The Weyl group symmetry introduced in this article is isomorphic to $\widetilde{W}(A_1^{(1)})$, which dose not seems to be a subgroup of the $\widetilde{W}(A_2^{(1)})$-symmetry discussed in [NY2, O]. To understand the relationship of our $\widetilde{W}(A_1^{(1)})$-symmetry to whole symmetry of $\mathrm{P_{IV}}$, it seems that we need to consider a larger group that contain both $\widetilde{W}(A_1^{(1)})$ and $\widetilde{W}(A_2^{(1)})$ as individual subgroups.

Though we limited ourselves to the $A_1^{(1)}$-case in this paper, our method may be applied to other type of affine Lie groups. For instance, in the case of $A_2^{(1)}$ non-standard hierarchy [KIK], we can obtain the fifth Painlevé equation with full parameters as a similarity reduction of the modified Yajima-Oikawa equation. We will dicuss this subject elsewhere.

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